Lecture 1C: Induction

UC Berkeley EECS 70
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Announcements!

~4pm lecture will be up

- Lecture is posted under "Media Gallery" in bCourses
- **HW 1** and **Vitamin 1** have been released, due **Today** (grace period Friday)

question 9

What is induction?

Goal in induction is to prove some statement for all natural numbers

$$(\forall n \in \mathbb{N}), P(n)$$

Principle of Induction

- Base Case: **Prove P(0)**
- Inductive Hypothesis: **Assume P(n)**
- Inductive Step: **Prove P(n) ⇒ P(n+1)**

these get combined

Direct Proof $P \Rightarrow Q$

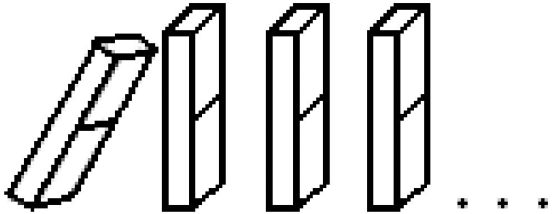
Assume P → Q

Visual Analogy

$P(5)$

$5 \le k$

Prove all the dominos fall down



- $P(0)$ = "First domino falls" Base Case
- $P(k) \Rightarrow P(k+1)$ "kth domino falls implies that k+1st domino falls" Inductive step

arbitrary

$\forall k \in \mathbb{N}$

Even if you had infinite* dominos lined up, this method would prove all of them will fall down (More on this Week 4).

Countability

Simple Induction (Example 1)

Base Case
Inductive Hypothesis
Inductive Step

Theorem: For all natural numbers n, $0 + 1 + 2 + ... + n = \frac{n(n+1)}{2}$
Proof:

Base Case: $n=0$ $\quad 0 = \frac{0(0+1)}{2} = 0$ ✓

Ind. Hyp.: Assume for some $n=k \geq 0$ it is true that $0+1+\dots+k = \frac{k(k+1)}{2}$

Ind. Step: Prove that for $n=k+1$ the claim holds

$$1+2+\dots+(k+1) = \frac{(k+1)(k+2)}{2}$$

$$\left[\underbrace{1+2+\dots+k}+(k+1) = \underbrace{\frac{k(k+1)}{2}} + (k+1)\right] = \frac{k^2+k+2k+2}{2} = \frac{(k+1)(k+2)}{2}$$

The second equality holds from the inductive hypothesis. Thus, the theorem holds by induction.

Simple Induction (Example 2)

Theorem: For all $n \in \mathbb{N}$, $3 | (n^3 - n)$

Proof:

We induct on the variable n

Base Case: $n=0$ $\quad 3 | 0^3 - 0$. This is trivially true.

Ind. Hyp: For $n=k$ assume $3 | k^3 - k$ i.e. $\exists q$ s.t. $\boxed{k^3 - k} = 3q$

Ind. Step: We wish to show that for $n = k+1$ $\quad 3 | (k+1)^3 - (k+1)$

$(k+1)^3 - (k+1) = 3p \qquad p \in \mathbb{N}$

$k^3 + 3k^2 + 3k + 1 - (k+1) = 3p$

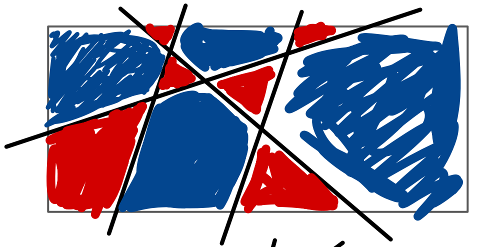
$k^3 - k + 3k^2 + 3k \cancel{+1} \cancel{-1} = 3p$

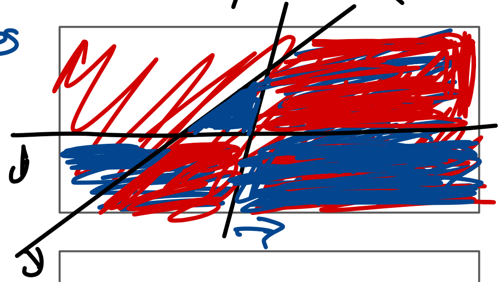
$3q + 3k^2 + 3k$ $\quad$ From the ind. hyp.

$3(\underbrace{q + k^2 + k}_{\in \mathbb{N}}) = 3p$ $\quad$ by def. it follows that $(k+1)^3 - (k+1)$ is divisible by 3 $\square$

$p = q + k^2 + k$

Simple Induction (Example 3)





Theorem: Any map formed by dividing the plain into regions by drawing *straight* lines can be properly colored with two colors

Proof: We will induct on the number of lines. Let n # of lines

Base Case: n = 0 color the whole plain one color

Ind Hyp: For n=k lines assume 'A' is two colorable

Ind Step: Consider an arbitrary map with k+1 lines. Then, remove one line from the map. By ind. hyp. this new map with k lines is two colorable. Then, add back the line that was removed and flip all the colors on one side of the line.

By construction all the regions adjacent to the line that was added have different colors. Then, the region that was not flipped is correctly colored by hyp. That was flipped, is also two colored by hypothesis since we just changed the labelings.

Improving Induction Hypothesis (Example 1)

"strengthening"

Theorem: The sum of the first n odd numbers is a perfect square

Improved: The sum of the first n odd numbers is n^2

Proof:

Base Case $n=1$: $1 = 1^2$ ✓

Ind Hyp: Assume $\underbrace{1+3+5+\dots+(2k-1)}_{\text{first } k \text{ odds}} = k^2$

Ind Step: Wish to show

$$\underbrace{1+3+5+\dots+(2k-1)}_{= k^2} + (2k+1) = (k+1)^2$$

$$k^2 + 2k + 1 \quad \text{by hyp.}$$

$$(k+1)^2 = \checkmark$$

$\square$

$1 = 1^2$

$1+3 = 2^2$

$1+3+5 = 3^2$

$1+3+5+7 = 4^2$

$\vdots$

$n=k$

$1+3+\dots+(2k-1) = k^2$

$k^2 + 2k+1 = (k+1)^2$

Improving Induction Hypothesis (Example 2)

notes :

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$

Improved:

Proof:

What is Strong Induction?

Goal:

$\forall n \quad P(n)$

Principle of Strong Induction

- Base Case: **Prove P(0)**
- Inductive Hypothesis: **Assume P(0) and P(1) and … and P(n)**
- Inductive Step: **Prove P(0) and … and P(n) ⇒ P(n+1)**

$$P(0) \land P(1) \land \ldots \land P(n) \implies P(n+1)$$

Strong induction is implied by weak induction

Strong Induction (Example 1)

prime factorization

Theorem: Every natural number greater than 1 can be written as a product of one or more primes

Proof:

Base Case: $n=2$. 2 is prime so it's prime factorization is just 2

Ind. Hyp: Assume claim holds for all $1 < n \leq k$

Ind. Step: let $n = k+1$

Case 1: $k+1$ is prime. we are done

Case 2: $k+1$ is composite. Therefore, $\exists a, b \in \mathbb{N}$, $k+1 = a \cdot b$

Since $k+1 > 1 \Rightarrow 1 < a, b < k+1$. Then, by the ind. hyp. a and b can be written as a product of primes. Thus, $k+1$ can be written as a product of a and b's primes.

$\square$

Strong Induction with Multiple Base Cases (Example 2)

Theorem: For every natural number $n \geq 12$, it holds that $n = 4x + 5y$ for some $x, y \in \mathbb{N}$

Proof:

Base Cases $n = 12$ $\quad 12 = 4(3) + 5(0) \quad x = 3, y = 0$ ✓

CkT

$12 = 4(3) + 5(0)$

$13 = 4(2) + 5(1)$

$14 = 4(1) + 5(2)$

$15 = 4(0) + 5(3)$

Ind Hyp: Assume claim holds for all $12 \leq n \leq k$

Ind Step: $n = k+1 \geq 16$. Then, $(k+1) - 4 \geq 12$

By the ind. hyp. $(k+1) - 4 = 4x' + 5y'$ for some $x', y' \in \mathbb{N}$

$k+1 = 4x' + 5y' + 4 = 4(x'+1) + 5y'$. So, then

we can set $x = x'+1$ and $y = y'$ $\quad \square$

$k+1 = 4x + 5y$

$n = 12$ ✓

$k = 4x + 5y$

$4(x+1) + 5(y+1)$

$k+1 = 4x' + 5y'$

$16 = 12 + 4$

$4x + 5y$

$4(x+1) + 5y$

$4x + 5y + 4$

$4(x+1) + 5y$,

Why ever use weak induction?

Weak Induction ⇒ Strong Induction

If you wanted to you could always use strong induction

It is nicer to only use weak induction if strong induction is not needed.

↳ it's easier for the reader

↳ easier to catch mistakes

Well-Ordering Principle

$S \subseteq \mathbb{N}$ and $S \neq \{\}$

$\exists n \in S$ s.t. $\forall m \in S$ $n \leq m$

The Well-Ordering Principle states that for any non-empty subset of the natural numbers there will be a least element.

has a prime factorization

Theorem: Every natural number greater than 1 can be written as a product of one or more primes

Proof using WOP:

Let S be the set of natural numbers that cannot be written as a product of primes. Assume for contradiction that S is not empty.

By WOP, S has a least element n.

o/w it would not be in S

Clearly, n is not prime. So, we can write $n = a \cdot b$ $a, b \in \mathbb{N}$.

It follows that a or b doesn't have a prime factorization.

Without loss of generality (WLOG) say a can't be written as a product of primes. Notice, since $n > 1$ $1 < a < n$. This is a contradiction because then $a \in S$, but we said n is the least element!

Thus, S is empty and theorem holds. $\square$

Summary

- Simple Induction
  - P(0) and show P(n) ⇒ P(n+1)
- Multiple Base Cases
  - You may need multiple base cases to prove a statement
- Improve the Inductive Hypothesis
  - Sometimes proving a "stronger" statement is easier
- Strong Induction
  - P(0) and show P(0) and … and P(n) ⇒ P(n+1)
- Well Ordering Principle
  - For any subset of the naturals there is a least element